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On the Recurrence Relations for *B*-Splines Defined by Certain *L*-Splines

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This paper provides a general unified recurrence relation for a class of *B*-splines defined by certain constant coefficient operators. The associated relations for polynomial, trigonometric, hyperbolic, and some particular Tchebycheffian *B*-splines are all covered as special cases. © 1985 Academic Press, Inc.

INTRODUCTION

The recurrence relations for polynomial, trigonometric, and hyperbolic *B*-splines were discovered separately in [1], [3], and [6, 7], respectively. They provide stable and efficient algorithms for computing with *B*-splines. It is desirable to establish analogous relations for other generalized *B*-splines. However, Schumaker pointed out in [6] that the above-mentioned spline spaces are essentially the only ones whose corresponding *B*-splines satisfy a recurrence relation similar to the polynomial case.

The purpose of this paper is to show that there exists a general *unified recurrence relation* for a class of *B*-splines, defined by certain constant coefficient differential operators. In particular, this general formula includes as special cases the recurrence relation for the three spline spaces mentioned above.

The crux of obtaining such a general recurrence relation is to discover a simple recurrence relation for the Green's functions of certain constant coefficient differential operators. To this end, we adopt a slightly different definition of generalized divided differences which is more suitable for our purpose.

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1. Green's Functions of Constant Coefficient Differential Operators

Given $l(\lambda)=(\lambda-\lambda_1)\cdots(\lambda-\lambda_m)$, where $\lambda_1,\dots,\lambda_m$ are arbitrary complex numbers. The theory of L-splines corresponding to the differential operator $l(D)$ was developed in [4], see also [2, 5]. We will freely draw upon results concerning these spline spaces. First we require a formula for the Green's function of the initial value problem associated with $l(D)$.

Let $G(x, t)$ be the Green's function of the operator $l(D)$ such that

$$l(D)\, G(x, t)=\delta(x-t) \tag{1.1}$$

$$G_x^{(j)}(a, t)=0, \qquad j=0, 1,\dots, m-1, \tag{1.2}$$

It is well known that if $H(x)$ is the solution of the homogeneous equations

$$l(D)\, H(x)=0 \tag{1.3}$$

$$H^{(j)}(0)=\delta_{m-1,j}, \qquad j=0, 1,\dots, m-1, \tag{1.4}$$

then $G(x, t)=H(x-t)(x-t)_+^0$.

Lemma 1. *The solution of problem* (1.3)–(1.4) *is*

$$H(x)=[\lambda_1,\dots,\lambda_m]\, e^{\lambda x}=[\lambda_1,\dots,\lambda_m]\, e^{\cdot x}. \tag{1.5}$$

Remark 1. Here the divided difference is taken with respect to λ. Consequently, the Green's function of (1.1)–(1.2) is

$$G(x, t)=[\lambda_1,\dots,\lambda_m]\, e^{\cdot(x-t)}(x-t)_+^0. \tag{1.6}$$

Proof. Differentiating (1.5) with respect to x, and using the well-known properties of ordinary divided difference we have

$$H^{(j)}(0)=[\lambda_1,\dots,\lambda_m]\, \lambda^j=\delta_{m-1,j}.$$

The lemma is proved. ∎

Lemma 2. *Suppose that*

$$\lambda_{j+1}-\lambda_j=\omega, \qquad j=1,\dots, m-1, \tag{1.7}$$

where ω is a constant, real or complex. Then

$$H(x) = e^{\lambda_1 x} \left(\frac{e^{\omega x} - 1}{\omega} \right)^{m-1} \Big/ (m-1)! \tag{1.8}$$

$$G(x, t) = e^{\lambda_1 (x-t)} \left(\frac{e^{\omega(x-t)} - 1}{\omega} \right)^{m-1} (x-t)_+^0 / (m-1)!. \tag{1.9}$$

Proof. Substituting (1.7) into (1.5) and calculating directly we get (1.8) and then (1.9). A more straightforward proof is to check that $H(x)$ in (1.8) satisfies (1.3) and (1.4). Since $H(x)$ is a linear combination of $e^{\lambda_1 x}, \ldots, e^{\lambda_m x}$, it satisfies (1.3); as to the condition (1.4), it is very easy to verify that it is also satisfied by $H(x)$.

LEMMA 3. *Under the condition* (1.7), *let*

$$\lambda^* = (\lambda_1 + \cdots + \lambda_m)/m = \lambda_1 + (m-1)\,\omega/2. \tag{1.10}$$

Then

$$H(x) = \left(\frac{2}{\omega} \right)^{m-1} e^{\lambda^* x} \left(\operatorname{sh} \frac{\omega}{2} x \right)^{m-1} \Big/ (m-1)! \tag{1.11}$$

$$G(x, t) = \left(\frac{2}{\omega} \right)^{m-1} e^{\lambda^*(x-t)} \left(\operatorname{sh} \frac{\omega}{2} (x-t) \right)^{m-1} (x-t)_+^0 / (m-1)!. \tag{1.12}$$

Proof. Since

$$\frac{e^{\omega x} - 1}{\omega} = \left(\frac{2}{\omega} \right) e^{(\omega/2)x} \left(e^{(\omega/2)x} - e^{-(\omega/2)x} \right)/2 = \left(\frac{2}{\omega} \right) e^{(\omega/2)x} \left(sh \frac{\omega}{2} x \right),$$

substituting this expression in (1.8) we get (1.11) immediately.

Remark 2. If $\omega = \sqrt{-1}$ and $\lambda^* = 0$ then

$$H(x) = 2^{m-1} \left(\sin \frac{x}{2} \right)^{m-1} \Big/ (m-1)!$$

$$G(x, t) = 2^{m-1} \left(\sin \frac{1}{2} (x-t) \right)^{m-1} (x-t)_+^0 / (m-1)!.$$

2. GENERALIZED DIVIDED DIFFERENCES

The definition of generalized divided differences corresponding to $l(D)$ is well known. For the purpose of deriving the recurrence relations for generalized B-splines, we modify the definition of this divided difference in the following way.

DEFINITION 1. Given an mth-order operator $l(D)=(D-\lambda_1)\cdots(D-\lambda_m)$ and a sequence of knots $t_1 \leqslant t_2 \leqslant \cdots \leqslant t_{m+1}$, the mth-order generalized divided difference with respect to the operator $l(D)$ and the knots $\{t_i\}^{m+1}$ is defined as the linear functional,

$$[t_1,..., t_{m+1}|l(D)]\, f = \sum_{j=1}^{m+1} a_j f^{(\alpha_j)}(t_j) \tag{2.1}$$

such that

$$[t_1,..., t_{m+1}|l(D)]\, f = 0 \qquad \text{whenever } l(D)\, f = 0, \tag{2.2}$$

where

$$\alpha_j = \max_{0 \leqslant r \leqslant j-1} \{r | t_{j-r} = t_j\}.$$

In (2.1) there are $m+1$ undetermined coefficients $\{a_j\}^{m+1}$ but the number of conditions in (2.2) is m, so the divided difference is not defined uniquely.

LEMMA 4. *Suppose that the condition* (1.7) *is satisfied, and*

$$t_1 < t_2 < \cdots < t_{m+1}$$

$$\operatorname{Re} \omega \neq 0, \text{ or } \operatorname{Re} \omega = 0 \text{ but } \omega(t_\mu - t_\nu) \neq 0 (\operatorname{mod} 2\pi),\ \mu \neq \nu. \tag{2.3}$$

Then

$$[t_1,..., t_{m+1}|l(D)]\, f = c \sum_{\mu=1}^{m+1} \omega^m e^{-\lambda_1 t_\mu} f(t_\mu) \Big/ \prod_{\nu \neq \mu} (e^{\omega t_\mu} - e^{\omega t_\nu}) \tag{2.4}$$

is the unique solution of (2.2) *where c is any constant which can be chosen arbitrarily.*

Proof. The denominator of (2.4) does not vanish, due to the condition (2.3). Let $y = e^{\omega t}$, $y_\mu = e^{\omega t_\mu}$, and $f(t) = e^{\lambda_j t}$. Then (2.4) becomes

$$\begin{aligned}[t_1,..., t_{m+1}|l(D)]\, f &= c\, \omega^m \sum_{\mu=1}^{m+1} y_\mu^{j-1} \Big/ \prod_{\nu \neq \mu} (y_\mu - y_\nu) \\ &= c\, \omega^m\, \delta_{m+1,j}.\end{aligned}$$

So the requirement (2.2) is satisfied. The uniqueness of (2.4) since any

$m \times m$ minor of the matrix corresponding to Eqs. (2.2) is a Vandermonde which is nonzero by condition (2.3). The proof is complete. ∎

Remark 3. Let $\lambda_1, \omega \to 0$, and take $c = 1$. Since

$$\frac{\omega}{e^{\omega t_\mu} - e^{\omega t_\nu}} \to \frac{1}{t_\mu - t_\nu}$$

(2.4) becomes the ordinary divided difference

$$[t_1,..., t_{m+1}] f = \sum_{\mu=1}^{m+1} f(t_\mu) \Big/ \prod_{\nu \neq \mu} (t_\mu - t_\nu).$$

For this reason we will choose $c = 1$ in Lemma 4 as our normalization for the generalized divided difference.

If $f(x)$ is sufficiently differentiable in $[a, b]$, then it is easy to verify $[t_1,..., t_{m+1} | l(D)] f$ is a continuous function of $t_1,..., t_{m+1}$. Thus if some of the knots become equal, the divided difference with multiple knots can be defined as the limit of those with simple knots.

LEMMA 5. *Under the conditions of Lemma* 3, *the divided difference* (2.4) *can be written in the form*

$$[t_1,..., t_{m+1} | l(D)] f = c' \sum_{\mu=1}^{m+1} \left(\frac{\omega}{2}\right)^m e^{-\lambda^* t_\mu} f(t_\mu) \Big/ \prod_{\nu \neq \mu} \operatorname{sh} \frac{\omega}{2} (t_\mu - t_\nu)$$

where

$$c' = e^{((m+1)/2)\omega t^*}$$

and

$$t^* = \sum_{\mu=1}^{m+1} t_\mu / (m+1).$$

Proof. Since

$$\sum_{\substack{\nu \neq \mu \\ \nu = 1}}^{m+1} (e^{\omega t_\mu} - e^{\omega t_\nu}) = \prod_{\nu \neq \mu} e^{(\omega/2)(t_\mu + t_\nu)} (e^{(\omega/2)(t_\mu - t_\nu)} - e^{-(\omega/2)(t_\mu - t_\nu)})$$

$$= 2^m e^{(\omega/2)(m-1) t_\mu} e^{(\omega/2)(m+1) t^*} \prod_{\nu \neq \mu} \operatorname{sh} \frac{\omega}{2} (t_\mu - t_\nu).$$

Substituting this expression into (2.4) completes the proof. ∎

3. *B*-Splines Defined by Certain Constant Coefficient Operators and Its Recurrence Relations

Given a real sequence $\{\xi_j\}^{n+m}$ satisfying

$$\xi_j \leqslant \xi_{j+1}, \qquad \xi_j < \xi_{j+m}.$$

DEFINITION 2. Suppose $\lambda_{j+1} - \lambda_j = \omega$, $j = 1,..., m-1$, then the mth-order B-splines associated with the operator $l_m(D) = (D - \lambda_1) \cdots (D - \lambda_m)$ and knots $\xi_j,..., \xi_{j+m}$ are defined as

$$B_{j,m}(x) = [\xi_j,..., \xi_{j+m} | l_m(-D)]\, K_m(x, \cdot) \tag{3.1}$$

where $K_m(x, t) = (m-1)!\, G(x, t)$.

We use the convention that $B_{j,m-1}$ is the B-spline with knots $\xi_j,...,$ ξ_{j+m-1} corresponding to the operator $l_{m-1}(D) = (D - \lambda_1) \cdots (D - \lambda_{m-1})$.

THEOREM 1. *Suppose that $m \geqslant 2$. Then we have*

$$B_{j,m}(x) = \frac{(e^{\omega x} - e^{\omega \xi_j})\, B_{j,m-1}(x) + (e^{\omega \xi_{j+m}} - e^{\omega x})\, B_{j+1,m-1}(x)}{e^{\omega \xi_{j+m}} - e^{\omega \xi_j}}. \tag{3.2}$$

Proof. For simplicity, we denote

$$[\xi_j,..., \xi_{j+m} | l_m(-D)]\, f = [\xi_j,..., \xi_{j+m}]^* f, \qquad \text{if } c = 1. \tag{3.3}$$

Suppose $\xi_1 < \cdots < \xi_{m+1}$ and let $y = e^{\omega t}$, $y_\mu = e^{\omega \xi_\mu}$. Since $K_m(x, t) =$ $K_{m-1}(x, t)(e^{\omega(x-t)} - 1)/\omega$ then by Definition 2, we have

$$\begin{aligned}
B_{1,m}(x) &= [\xi_1,..., \xi_{n+1}]^* K_m(x, \cdot) \\
&= (-1)^m \omega^m \sum_{\mu=1}^{m+1} e^{\lambda_m \xi_\mu} K_m(x, \xi_\mu) \Big/ \prod_{v \neq \mu} (y_\mu - y_v) \\
&= (-1)^m \omega^{m-1} \sum_{\mu=1}^{m+1} e^{\lambda_m \xi_\mu} (e^{\omega(x-\xi_\mu)} - 1)\, K_{m-1}(x, \cdot) \Big/ \prod_{v \neq \mu} (y_\mu - y_v) \\
&= (-1)^m \omega^{m-1} \{(e^{\omega x} - y_1)[y_1,..., y_{m+1}]\, e^{\lambda_{m-1} \cdot} K_{m-1}(x, \cdot) \\
&\quad - [y_2,..., y_{m+1}]\, e^{\lambda_{m-1} \cdot} K_{m-1}(x, \cdot)\} \\
&= \frac{1}{y_{m+1} - y_1} \{(e^{\omega x} - e^{\omega \xi_1})\, B_{1,m-1}(x) + (e^{\omega \xi_{m+1}} - e^{\omega x})\, B_{2,m-1}(x)\}.
\end{aligned}$$

This is just (3.2) for $j = 1$. The theorem is proved.

Next we present another recurrence relation which requires different conditions on $\lambda_1,..., \lambda_m$ than in Theorem 1. In our next result, we do not require that the zeros of $l_{m-1}(D)$ are also zeros of $l_m(D)$. Instead we only demand that the averages of the zeros of $l_m(D)$ and $l_{m-1}(D)$ are the same. To this end, it is convenient to renormalize the B-splines and introduce

$$\langle \xi_j,..., \xi_{j+m} \rangle^* = e^{((m+1)/2)\omega\xi^*} [\xi_j,..., \xi_{j+m}]^*$$

and

$$M_{j,m} = e^{((m+1)/2)\omega\xi_j^*} B_{j,m} = \langle \xi_j,..., \xi_{j+m} \rangle^* K_m(x, \cdot)$$

where

$$\xi_j^* = \frac{1}{m+1} \sum_{k=0}^{m} \xi_{j+k}.$$

THEOREM 2. *Suppose that $m \geqslant 2$ and the polynomials $l_r(\lambda) = (\lambda - \lambda_1^{(r)}) \cdots (\lambda - \lambda_m^{(r)})$, $r = m-1, m$, are such that*

$$\lambda_{\mu+1}^{(r)} - \lambda_\mu^{(r)} = \omega, \qquad \mu = 1,..., r-1, \tag{3.4}$$

$$\sum_{\mu=1}^{r} \lambda_\mu^{(r)} = r\lambda^*. \tag{3.5}$$

Then there exists an analogous recurrence relation for the B-splines defined by $l_r(D)$, $r = m-1, m$, given by

$$M_{j,m}(x) = \frac{\operatorname{sh}\frac{\omega}{2}(x-\xi_j)\, M_{j,m-1}(x) + \operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-x)\, M_{j+1,m-1}(x)}{\operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-\xi_j)}. \tag{3.6}$$

In order to prove Theorem 2, we need

LEMMA 6. *We have the following two identities*:

$$\begin{aligned}\langle \xi_1,..., \xi_{m+1} \rangle^* &\left(f(\cdot) \frac{2}{\omega} \operatorname{sh}\frac{\omega}{2}(x - \cdot) \right) \\ &= \frac{1}{\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-\xi_1)} \left\{ \operatorname{sh}\frac{\omega}{2}(x-\xi_1) \langle \xi_1,..., \xi_m \rangle^* f \right. \\ &\qquad \left. + \operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-x) \langle \xi_2,..., \xi_{m+1} \rangle^* f \right\}\end{aligned} \tag{3.7}$$

and

$$\langle \xi_1,\dots,\xi_{m+1}\rangle^*\left(\operatorname{ch}\frac{\omega}{2}(x-\cdot)f(\cdot)\right)$$

$$=\frac{\omega}{2\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-\xi_1)}\left\{\operatorname{ch}\frac{\omega}{2}(x-\cdot)\langle \xi_1,\dots,\xi_m\rangle^*\right.$$

$$\left.+\operatorname{ch}\frac{\omega}{2}(\xi_{j+m}-x)\langle \xi_2,\dots,\xi_{j+m}\rangle^* f\right\}. \tag{3.8}$$

Proof. By Definition 2, if $\xi_1<\cdots<\xi_{m+1}$ then

$$\langle \xi_1,\dots,\xi_{m+1}\rangle^*\left(\frac{2}{\omega}\operatorname{sh}\frac{\omega}{2}(x-\cdot)f(\cdot)\right)$$

$$=(-1)^m\sum_{\mu=1}^{m+1}\left(\frac{\omega}{2}\right)^{m-1}e^{\lambda^*\xi_\mu}\operatorname{sh}\frac{\omega}{2}(x-\xi_\mu)f(\xi_\mu)\Big/\prod_{\nu\neq\mu}\operatorname{sh}\frac{\omega}{2}(\xi_\mu-\xi_\nu)$$

$$\operatorname{sh}\frac{\omega}{2}(x-\xi_1)\langle \xi_1,\dots,\xi_m\rangle^* f$$

$$=(-1)^m\sum_{\mu=1}^{m+1}\left(\frac{\omega}{2}\right)^{m-1}e^{\lambda^*\xi_\mu}\operatorname{sh}\frac{\omega}{2}(x-\xi_\mu)f(\xi_\mu)\Big/\prod_{\nu\neq\mu}\operatorname{sh}\frac{\omega}{2}(\xi_\mu-\xi_\nu)$$

$$\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-x)\langle \xi_2,\dots,\xi_{m+1}\rangle^* f$$

$$=(-1)^m\sum_{\mu=1}^{m+1}\left(\frac{\omega}{2}\right)^{m-1}e^{\lambda^*\xi_\mu}\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-x)f(\xi_\mu)\Big/\sum_{\nu\neq\mu}\operatorname{sh}\frac{\omega}{2}(\xi_\mu-\xi_\nu).$$

To show the equality of the two sides of (3.7), it suffices to compare the coefficients of $f(\xi_1),\dots,f(\xi_{m+1})$. It is easy to see the coefficients of $f(\xi_1)$ and $f(\xi_{m+1})$ agree. For $1<\mu<m+1$ they also agree since we have

$$\operatorname{sh}\frac{\omega}{2}(x-\xi_1)\operatorname{sh}\frac{\omega}{2}(\xi_\mu-\xi_{m+1})+\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-x)\operatorname{sh}\frac{\omega}{2}(\xi_\mu-\xi_1)$$

$$=-\operatorname{sh}\frac{\omega}{2}(x-\xi_\mu)\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-x)$$

which follows from

$$\operatorname{sh}\frac{\omega}{2}(x-\xi_1)=\operatorname{sh}\frac{\omega}{2}(x-\xi_\mu)\operatorname{ch}\frac{\omega}{2}(\xi_\mu-\xi_1)+\operatorname{ch}\frac{\omega}{2}(x-\xi_\mu)\operatorname{sh}\frac{\omega}{2}(\xi_\mu-\xi_1)$$

and

$$\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-x)=\operatorname{sh}\frac{\omega}{2}(\xi_{m+1}-\xi_\mu)\operatorname{ch}\frac{\omega}{2}(\xi_\mu-x)$$
$$+\operatorname{ch}\frac{\omega}{2}(\xi_{m+1}-\xi_\mu)\operatorname{sh}\frac{\omega}{2}(\xi_\mu-x).$$

This shows that (3.7) holds for distinct ξ's. The fact that it is also valid for general $\xi_1\leqslant\xi_2\leqslant\cdots\leqslant\xi_{m+1}$ follows by the continuity of the divided differences.

The proof of (3.8) is similar.

If $\omega=2$, then Lemma 6 reduces to Theorem 4.2 in [6]. The proof here is just like that of Theorem 4.2.

Proof of Theorem 2. Since

$$K_m(x,t)=\left(\frac{2}{\omega}\right)\operatorname{sh}\frac{\omega}{2}(x-t)\,K_{m-1}(x,t),$$

if $\xi_1<\xi_2<\cdots<\xi_{m+1}$, by Lemma 6 we have

$$\begin{aligned}
M_{j,m}(x)&=\langle\xi_j,\ldots,\xi_{j+m}\rangle^*\,K_m(x,\cdot)\\
&=\langle\xi_j,\ldots,\xi_{j+m}\rangle^*\left(\frac{2}{\omega}\operatorname{sh}\frac{\omega}{2}(x-\cdot)\right)K_{m-1}(x,\cdot)\\
&=\frac{1}{\operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-\xi_j)}\\
&\quad\left\{\operatorname{sh}\frac{\omega}{2}(x-\xi_j)\langle\xi_j,\ldots,\xi_{j+m-1}\rangle^*\,K_{m-1}(x,\cdot)\right.\\
&\quad\left.+\operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-x)\langle\xi_{j+1},\ldots,\xi_{j+m}\rangle^*\,K_{m-1}(x,\cdot)\right\}\\
&=\frac{1}{\operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-\xi_j)}\left\{\operatorname{sh}\frac{\omega}{2}(x-\xi_j)\,M_{j,m-1}(x)\right.\\
&\quad\left.+\operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-x)\,M_{j+1,m-1}(x)\right\}.
\end{aligned}$$

This shows that Theorem 2 holds for distinct ξ's. For arbitrary ξ's it follows by the continuity of the divided differences. The proof of Theorem 2 is complete.

THEOREM 3. *Let $m \geqslant 2$. Then for the derivatives of B-splines in Theorem* 1 we have

$$B'_{j,m}(x) = \lambda_1 B_{j,m}(x) + (m-1)\,\omega e^{\omega x}\left(\frac{B_{j,m-1}(x) - B_{j+1,m-1}(x)}{e^{\omega\xi_{j+m}} - e^{\omega\xi_j}}\right). \quad (3.9)$$

Proof. Since

$$B'_{j,m}(x) = [\xi_j, ..., \xi_{j+m}]^*\, K'_m(x, \cdot) \quad (3.10)$$

and

$$K'_m(x, \cdot) = \lambda_1 K_m(x, \cdot) + (m-1)\, e^{\omega(x-\cdot)} K_{m-1}(x, \cdot). \quad (3.11)$$

Applying the operator $[\xi_j, ..., \xi_{j+m}]^*$ to the two sides of (3.11) and substituting the result into (3.10), (3.9) is obtained immediately.

THEOREM 4. *Let $m \geqslant 2$. Then for the derivatives of B-splines in Theorem 2 we have*

$$M'_{j,m}(x) = \lambda^* M_{j,m}(x) + \frac{(m-1)\,\omega}{2} \times \left\{\frac{\operatorname{ch}\frac{\omega}{2}(x-\xi_j)\, M_{j,m-1}(x) - \operatorname{ch}\frac{\omega}{2}(\xi_{j+m}-x)\, M_{j+1,m-1}(x)}{\operatorname{sh}\frac{\omega}{2}(\xi_{j+m}-\xi_j)}\right\}. \quad (3.12)$$

Proof. Since

$$K'_m(x, t) = \lambda^* K'_m(x, t) + (m-1)\operatorname{ch}\frac{\omega}{2}(x-t)\, K_{m-1}(x, t)$$

here the derivative is taken with respect to x. Applying the operator $\langle \xi_j, ..., \xi_{j+m}\rangle^*$ to both sides and using (3.8), we get (3.12) immediately.

COROLLARY 1. *In Theorem* 1 *and* 3 *let $\omega, \lambda_1 \to 0$. Then the associated relations for polynomial B-splines are obtained.*

COROLLARY 2. *Let*

$$l_1(D)=D$$

$$l_2(D)=\left(D-\frac{\omega}{2}\right)\left(D+\frac{\omega}{2}\right)$$

$$\cdots$$

$$l_{2r}(D)=\left(D^2-\left(\frac{\omega}{2}\right)^2\right)\left(D^2-\left(\frac{3}{2}\omega\right)^2\right)\cdots\left(D^2-\left(r-\frac{1}{2}\right)^2\omega^2\right)$$

$$l_{2r+1}(D)=D(D^2-\omega^2)\ldots\ldots(D^2-r^2\omega^2).$$

By choosing $\omega=2$ and $\omega=i$, respectively, the hyperbolic and trigometric B-spline recurrence relations in [7, 3] are obtained from Theorems 2 and 4.

Remark 4. It is worthwhile to notice that by Definition 2, for $l_1(D)=(D-\lambda_1)$ the *B*-splines are

$$B_{j,1}(x)=\omega\left(\frac{e^{\lambda_1 x}}{e^{\omega\xi_{j+1}}-d^{\omega\xi_j}}\right)((x-\xi_j)^0_+-(x-\xi_{j+1})^0_+)\qquad(c=1).$$

If $\lambda_1\neq 0$ they do not belong to the class of *B*-splines defined in Algorithms 3.2 and 3.3 [6], but they are particular Tchebycheffian splines considered in [6].

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